

# MODULE VARIETIES AND REPRESENTATION TYPE OF FINITE-DIMENSIONAL ALGEBRAS

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**ABSTRACT.** In this paper we seek invariant-theoretic characterizations of (Schur-)representation finite algebras. To this end, we introduce two classes of finite-dimensional algebras: those with the dense-orbit property and those with the multiplicity-free property. We show first that when a connected algebra  $A$  admits a pre-projective component, each of these properties is equivalent to  $A$  being representation-finite. Next, we give an example of a representation-infinite algebra with the dense-orbit property. We also show that the string algebras with the dense orbit-property are precisely the representation-finite ones. Finally, we show that a tame algebra has the multiplicity-free property if and only if it is Schur-representation-finite.

## CONTENTS

1. Introduction	1
2. Background	3
3. Representation-infinite DO algebras	9
4. Representation-infinite MF algebras	15
5. Open questions on DO algebras	18
6. Open questions on MF algebras	20
References	21

## 1. INTRODUCTION

Throughout the article,  $K$  always denotes an algebraically closed field of characteristic zero, and “algebra” refers to an associative  $K$ -algebra with identity. (We will remark at times on particular results that hold in arbitrary characteristic.) All modules are assumed to be finite-dimensional left modules. We use interchangeably the vocabulary of modules over finite-dimensional algebras, and that of representations of quivers with relations. A summary of the background on these, and their varieties of modules/representations, is given in Section 2.

The questions we explore are inspired by the following well-known theorem, whose proof is outlined below.

**Theorem.** *A quiver  $Q$  admits only finitely many indecomposable representations if and only if, for all dimension vectors  $\mathbf{d}$ , the base change group  $\mathrm{GL}(\mathbf{d})$  acts on  $\mathrm{mod}(Q, \mathbf{d})$  with a dense orbit.*

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The reasoning is straightforward: if  $Q$  admits only finitely many indecomposables, then it admits only finitely many isomorphism classes of representations of a given dimension vector. Whenever there are finitely many  $\mathrm{GL}(\mathbf{d})$ -orbits in some  $\mathrm{mod}(Q, \mathbf{d})$ , then since  $\mathrm{mod}(Q, \mathbf{d})$  is an irreducible variety, it must have a dense orbit. On the other hand, if  $Q$  is of affine Dynkin type, then any imaginary root  $\mathbf{d}$  gives a dimension vector for which  $\mathrm{mod}(Q, \mathbf{d})$  does not have a dense  $\mathrm{GL}(\mathbf{d})$ -orbit. So if  $Q$  is a quiver such that all  $\mathrm{mod}(Q, \mathbf{d})$  have a dense  $\mathrm{GL}(\mathbf{d})$ -orbit, then it cannot contain any subquiver which is affine Dynkin. Thus  $Q$  must be Dynkin and therefore of finite representation type by Gabriel's Theorem.

We wish to explore how this may be generalized to quivers with relations, or equivalently, to arbitrary finite-dimensional algebras. We are thus lead to study bound quiver algebras with the property that each irreducible component of each of their module varieties has a dense orbit. This dense orbit property that we want to utilize implies that all weight spaces of semi-invariants have dimension at most one. The hope was that the latter condition would imply finite representation type. It turns out that this can be achieved only for certain classes of algebras:

**Theorem 1.** *Let  $A$  be a connected, bound quiver algebra with a preprojective component. Then, the following properties are equivalent:*

- (1)  *$A$  is representation-finite;*
- (2) *for each dimension vector  $\mathbf{d}$  of  $A$ ,  $\mathrm{GL}(\mathbf{d})$  acts on each irreducible component of  $\mathrm{mod}(A, \mathbf{d})$  with a dense orbit;*
- (3) *for each dimension vector  $\mathbf{d}$  of  $A$  and each irreducible component  $C$  of  $\mathrm{mod}(A, \mathbf{d})$ , the algebra of semi-invariants  $K[C]^{\mathrm{SL}(\mathbf{d})}$  is multiplicity-free.*

A bound quiver algebra  $A$  with property (2) above is said to have *the dense orbit property* (write  $A$  is DO). If  $A$  satisfies property (3) above, we say it has *the multiplicity-free property* (write  $A$  is MF).

Theorem 1 is no longer true when the connected algebra in question does not have a preprojective component. The following Theorem shows that (2) does not imply (1) for a general finite-dimensional algebra. Its proof comprises Section 3.

**Theorem 2.** *Let  $A$  be the 2-point minimal representation-infinite algebra given by the following quiver and relations:*

$$\begin{array}{ccc} \sigma \curvearrowright \bullet & \xrightarrow{\nu} & \bullet \curvearrowright \rho \\ 1 & & 2 \end{array} \quad \sigma^4 = \rho^4 = \rho^2\nu = 0, \quad \nu\sigma = \rho\nu.$$

*Then,  $A$  is DO; in particular, it is MF.*

We should point out that the example above is of infinite global dimension; in fact, all of our examples of representation-infinite DO algebras are 2-point algebras of infinite global dimension.

It is easier to produce examples of algebras for which (3) does not imply (1); a small example is given in Section 4.1 and a complete analysis of the situation for string algebras is carried out in Section 4.2. However, if we replace “representation-finite” with “Schur-representation-finite” (see Section 2.1) then it becomes more plausible.

Finally, we prove (see Proposition 26 and Theorem 21):

**Theorem 3.** *Let  $A = KQ/I$  be a bound quiver algebra.*

- (1) Assume that  $A$  is a string algebra. Then,  $A$  is representation-finite if and only if  $A$  is  $DO$ .
- (2) Assume that  $A$  is tame. Then,  $A$  is Schur-representation-finite if and only if  $A$  is  $MF$ .

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## 2. BACKGROUND

**2.1. Bound quiver algebras.** Let  $Q = (Q_0, Q_1, t, h)$  be a finite quiver with vertex set  $Q_0$  and arrow set  $Q_1$ . The two functions  $t, h : Q_1 \rightarrow Q_0$  assign to each arrow  $a \in Q_1$  its tail  $ta$  and head  $ha$ , respectively.

A representation  $V$  of  $Q$  over  $K$  is a collection  $(V(i), V(a))_{i \in Q_0, a \in Q_1}$  of finite-dimensional  $K$ -vector spaces  $V(i)$ ,  $i \in Q_0$ , and  $K$ -linear maps  $V(a) \in \text{Hom}_K(V(ta), V(ha))$ ,  $a \in Q_1$ . The dimension vector of a representation  $V$  of  $Q$  is the function  $\mathbf{dim} V : Q_0 \rightarrow \mathbb{Z}$  defined by  $(\mathbf{dim} V)(i) = \dim_K V(i)$  for  $i \in Q_0$ . Let  $S_i$  be the one-dimensional representation of  $Q$  at vertex  $i \in Q_0$  and let us denote by  $\mathbf{e}_i$  its dimension vector. By a dimension vector of  $Q$ , we simply mean a function  $\mathbf{d} \in \mathbb{Z}_{\geq 0}^{Q_0}$ .

Given two representations  $V$  and  $W$  of  $Q$ , we define a morphism  $\varphi : V \rightarrow W$  to be a collection  $(\varphi(i))_{i \in Q_0}$  of  $K$ -linear maps with  $\varphi(i) \in \text{Hom}_K(V(i), W(i))$  for each  $i \in Q_0$ , and such that  $\varphi(ha)V(a) = W(a)\varphi(ta)$  for each  $a \in Q_1$ . We denote by  $\text{Hom}_Q(V, W)$  the  $K$ -vector space of all morphisms from  $V$  to  $W$ . Let  $V$  and  $W$  be two representations of  $Q$ . We say that  $V$  is a subrepresentation of  $W$  if  $V(i)$  is a subspace of  $W(i)$  for each  $i \in Q_0$  and  $V(a)$  is the restriction of  $W(a)$  to  $V(ta)$  for each  $a \in Q_1$ . In this way, we obtain the abelian category  $\text{rep}(Q)$  of all representations of  $Q$ .

Given a quiver  $Q$ , its path algebra  $KQ$  has a  $K$ -basis consisting of all paths (including the trivial ones) and the multiplication in  $KQ$  is given by concatenation of paths. It is easy to see that any  $KQ$ -module defines a representation of  $Q$ , and vice-versa. Furthermore, the category  $\text{mod}(KQ)$  of  $KQ$ -modules is equivalent to the category  $\text{rep}(Q)$ . In what follows, we identify  $\text{mod}(KQ)$  and  $\text{rep}(Q)$ , and use the same notation for a module and the corresponding representation.

A two-sided ideal  $I$  of  $KQ$  is said to be *admissible* if there exists an integer  $L \geq 2$  such that  $R_Q^L \subseteq I \subseteq R_Q^2$ . Here,  $R_Q$  denotes the two-sided ideal of  $KQ$  generated by all arrows of  $Q$ .

If  $I$  is an admissible ideal of  $KQ$ , the pair  $(Q, I)$  is called a *bound quiver* and the quotient algebra  $KQ/I$  is called the *bound quiver algebra* of  $(Q, I)$ . Any admissible ideal is generated by finitely many admissible relations, and any bound quiver algebra is finite-dimensional and basic. Moreover, a bound quiver algebra  $KQ/I$  is connected if and only if (the underlying graph of)  $Q$  is connected (see for example [ASS06, Lemma II.2.5]).

Up to Morita equivalence, any finite-dimensional algebra  $A$  can be viewed as the bound quiver algebra of a bound quiver  $(Q_A, I)$ , where  $Q_A$  is the Gabriel quiver of  $A$  (see [ASS06, Corollary I.6.10 and Theorem II.3.7]). (Note that the ideal of relations  $I$  is not uniquely determined by  $A$ .) We say that  $A$  is a *triangular* algebra if its Gabriel quiver has no oriented cycles.

Fix a bound quiver  $(Q, I)$  and let  $A = KQ/I$  be its bound quiver algebra. A representation  $M$  of a  $A$  (or  $(Q, I)$ ) is just a representation  $M$  of  $Q$  such that  $M(r) = 0$  for all  $r \in I$ . The category  $\text{mod}(A)$  of finite-dimensional left  $A$ -modules is equivalent to the category  $\text{rep}(A)$  of representations of  $A$ . As before, we identify  $\text{mod}(A)$  and  $\text{rep}(A)$ , and make no distinction between  $A$ -modules and representations of  $A$ . For each vertex  $x \in Q_0$ , we denote by  $P_x$  the projective indecomposable  $A$ -module at vertex  $x$ .

The algebra  $A$  is said to be *representation-finite* if there are finitely many indecomposable  $A$ -modules, up to isomorphism. According to the Brauer-Thrall II Conjecture, a theorem by now over perfect fields, a finite-dimensional algebra is representation-finite if and only if, for each dimension vector  $\mathbf{d}$  of  $A$ , there are only finitely many  $\mathbf{d}$ -dimensional indecomposable  $A$ -modules. We say that  $A$  is *tame* if for every dimension  $d$ , almost all  $d$ -dimensional indecomposable  $A$ -modules occur in finitely many 1-parameter families. The algebra  $A$  is said to be *wild* if its module theory is at least as complicated as that of the free algebra in two variables. For more precise definitions, we refer to [SS07b].

Finally, we say that  $A$  is *Schur-representation-finite* if, for each dimension vector  $\mathbf{d}$  of  $A$ , there are finitely many  $\mathbf{d}$ -dimensional Schur  $A$ -modules up to isomorphism. Recall that an  $A$ -module  $M$  is said to be *Schur* if  $\text{End}_A(M) \simeq K$ .

**2.2. Module varieties and the DO property.** Let  $\mathbf{d}$  be a dimension vector of  $A = KQ/I$  (or equivalently, of  $Q$ ). The affine variety

$$\text{mod}(A, \mathbf{d}) := \{M \in \prod_{a \in Q_1} \text{Mat}_{\mathbf{d}(ha) \times \mathbf{d}(ta)}(K) \mid M(r) = 0, \forall r \in I\}$$

is called the *module/representation variety* of  $\mathbf{d}$ -dimensional modules/representations of  $A$ . The affine space  $\text{mod}(Q, \mathbf{d}) := \prod_{a \in Q_1} \text{Mat}_{\mathbf{d}(ha) \times \mathbf{d}(ta)}(K)$  is acted upon by the base change group

$$\text{GL}(\mathbf{d}) := \prod_{i \in Q_0} \text{GL}(\mathbf{d}(i), K)$$

by simultaneous conjugation, i.e., for  $g = (g(i))_{i \in Q_0} \in \text{GL}(\mathbf{d})$  and  $V = (V(a))_{a \in Q_1} \in \text{mod}(Q, \mathbf{d})$ ,  $g \cdot V$  is defined by

$$(g \cdot V)(a) = g(ha)V(a)g(ta)^{-1}, \forall a \in Q_1.$$

It can be easily seen that  $\text{mod}(A, \mathbf{d})$  is a  $\text{GL}(\mathbf{d})$ -invariant closed subvariety of  $\text{mod}(Q, \mathbf{d})$ , and that the  $\text{GL}(\mathbf{d})$ -orbits in  $\text{mod}(A, \mathbf{d})$  are in one-to-one correspondence with the isomorphism classes of the  $\mathbf{d}$ -dimensional  $A$ -modules.

Note that  $\text{mod}(A, \mathbf{d})$  does not have to be irreducible. Let  $C$  be an irreducible component of  $\text{mod}(A, \mathbf{d})$ . We say that  $C$  is *indecomposable* if  $C$  has a non-empty open subset of indecomposable modules; whenever  $\text{mod}(A, \mathbf{d})$  has such an irreducible component, we say that  $\mathbf{d}$  is a *generic root* of  $A$ .

Now, let us consider a decomposition  $\mathbf{d} = \mathbf{d}_1 + \dots + \mathbf{d}_t$  where  $\mathbf{d}_i \in \mathbb{Z}_{\geq 0}^{Q_0}$ ,  $1 \leq i \leq t$ . Given a  $\text{GL}(\mathbf{d}_i)$ -invariant subset  $C_i \subseteq \text{mod}(A, \mathbf{d}_i)$ ,  $1 \leq i \leq t$ , we consider the following constructible subset of  $\text{mod}(A, \mathbf{d})$ :

$$C_1 \oplus \dots \oplus C_t = \{M \in \text{mod}(A, \mathbf{d}) \mid M \simeq \bigoplus_{i=1}^t M_i \text{ with } M_i \in C_i, \forall 1 \leq i \leq t\}.$$

As shown by de la Peña in [dlP91, §1.3] and Crawley-Boevey and Schröer in [CBS02, Theorem 1.1], any irreducible component of a module variety satisfies a Krull-Schmidt type decomposition. Specifically, if  $C$  is an irreducible component of  $\text{mod}(A, \mathbf{d})$  then there are unique generic roots  $\mathbf{d}_1, \dots, \mathbf{d}_t$  of  $A$  such that  $\mathbf{d} = \mathbf{d}_1 + \dots + \mathbf{d}_t$  and

$$C = \overline{C_1 \oplus \dots \oplus C_t}$$

for some indecomposable irreducible components  $C_i \subseteq \text{mod}(A, \mathbf{d}_i)$ ,  $1 \leq i \leq t$ . Moreover, the indecomposable irreducible components  $C_i$ ,  $1 \leq i \leq t$ , are uniquely determined by this property. We call  $\mathbf{d} = \mathbf{d}_1 \oplus \dots \oplus \mathbf{d}_t$  the *generic decomposition of  $\mathbf{d}$  in  $C$* , and  $C = \overline{C_1 \oplus \dots \oplus C_t}$  the *generic decomposition of  $C$* .

**Definition 4.** An algebra  $A = KQ/I$  is said to have the *dense orbit property* (write  $A$  is DO) if each irreducible component of each of its module varieties has a dense orbit.

*Remark 5.* Using the generic decomposition for irreducible components, it is easy to see that an algebra  $A = KQ/I$  is DO if and only if all of its indecomposable irreducible components are orbit closures.

*Remark 6.* Note that any DO algebra  $A = KQ/I$  is Schur-representation-finite. Indeed, if there is a dimension vector  $\mathbf{d}$  of  $A$  such that  $\text{mod}(A, \mathbf{d})$  contains infinitely many Schur  $A$ -modules then one of its irreducible components, say  $C = \overline{\text{GL}(\mathbf{d})M}$ , would contain a Schur  $A$ -module  $M' \not\cong M$ . But then  $1 = \dim_K \text{End}_A(M') > \dim_K \text{End}_A(M)$  which is, of course, impossible.

**2.3. Weight spaces of semi-invariants and the MF property.** Let  $\mathbf{d} \in \mathbb{Z}_{\geq 0}^{Q_0}$  be a dimension vector of  $A = KQ/I$ .

Since  $A$  is finite-dimensional, we know that there are only finitely many simple  $A$ -modules; in particular, there is a unique  $\mathbf{d}$ -dimensional semi-simple module in  $\text{mod}(A, \mathbf{d})$ . This is furthermore equivalent to saying that there is only one closed  $\text{GL}(\mathbf{d})$ -orbit in  $\text{mod}(A, \mathbf{d})$ , and hence the invariant ring  $\text{I}(A, \mathbf{d}) := K[\text{mod}(A, \mathbf{d})]^{\text{GL}(\mathbf{d})}$  is exactly the base field  $K$ .

Now, let us consider the subgroup  $\text{SL}(\mathbf{d}) \subseteq \text{GL}(\mathbf{d})$  defined by

$$\text{SL}(\mathbf{d}) = \prod_{i \in Q_0} \text{SL}(\mathbf{d}(i), K).$$

Although there are only constant  $\text{GL}(\mathbf{d})$ -invariant regular functions on  $\text{mod}(A, \mathbf{d})$ , the action of  $\text{SL}(\mathbf{d})$  on  $\text{mod}(A, \mathbf{d})$  provides us with a highly non-trivial ring of semi-invariants.

Note that any  $\theta \in \mathbb{Z}^{Q_0}$  defines a rational character  $\chi_\theta : \text{GL}(\mathbf{d}) \rightarrow K^*$  by

$$\chi_\theta((g(i))_{i \in Q_0}) = \prod_{i \in Q_0} (\det g(i))^{\theta(i)}.$$

In this way, we can identify  $\Gamma = \mathbb{Z}^{Q_0}$  with the group  $X^*(\text{GL}(\mathbf{d}))$  of rational characters of  $\text{GL}(\mathbf{d})$ , assuming that  $\mathbf{d}$  is a sincere dimension vector. In general, we have only the natural epimorphism  $\Gamma \rightarrow X^*(\text{GL}(\mathbf{d}))$ . We also refer to the rational characters of  $\text{GL}(\mathbf{d})$  as (integral) weights of  $A$  (or  $Q$ ).

Let us now consider the ring of semi-invariants  $\text{SI}(A, \mathbf{d}) := K[\text{mod}(A, \mathbf{d})]^{\text{SL}(\mathbf{d})}$ . As  $\text{SL}(\mathbf{d})$  is the commutator subgroup of  $\text{GL}(\mathbf{d})$  and  $\text{GL}(\mathbf{d})$  is linearly reductive, we have

$$\text{SI}(A, \mathbf{d}) = \bigoplus_{\theta \in X^*(\text{GL}(\mathbf{d}))} \text{SI}(A, \mathbf{d})_\theta,$$

where

$$\mathrm{SI}(A, \mathbf{d})_\theta = \{f \in K[\mathrm{mod}(A, \mathbf{d})] \mid gf = \theta(g)f \text{ for all } g \in \mathrm{GL}(\mathbf{d})\}$$

is called *the space of semi-invariants* on  $\mathrm{mod}(A, \mathbf{d})$  of weight  $\theta$ .

For an irreducible component  $C \subseteq \mathrm{mod}(A, \mathbf{d})$ , we similarly define the ring of semi-invariants  $\mathrm{SI}(C) := K[C]^{\mathrm{SL}(\mathbf{d})}$ , and the space  $\mathrm{SI}(C)_\theta$  of semi-invariants on  $C$  of weight  $\theta \in \mathbb{Z}^{Q_0}$ .

**Definition 7.** An algebra  $A = KQ/I$  is said to have the *multiplicity-free property* (write  $A$  is MF) if the algebra of semi-invariants over each irreducible component of each of its module varieties is multiplicity-free; that is, each weight space of semi-invariants has dimension at most one.

Our motivation for studying MF algebras in relation to DO algebras is given by the following simple lemma.

**Lemma 8.** *Let  $A$  be a bound quiver algebra and  $C$  a component of  $\mathrm{mod}(A, \mathbf{d})$  with a dense orbit. Then the coordinate ring  $K[C]$  is multiplicity-free. In particular, if  $A$  is DO, then  $A$  is MF.*

*Proof.* Write  $C = \overline{\mathrm{GL}(\mathbf{d})M_0}$  for some  $M_0 \in \mathrm{mod}(A, \mathbf{d})$ . Let  $\theta \in \mathbb{Z}^{Q_0}$  be an integral weight of  $A$  and let  $f, h \in \mathrm{SI}(C)_\theta$  be two non-zero semi-invariants of weight  $\theta$ . In particular, we know that  $f(M_0)$  and  $h(M_0)$  are non-zero scalars and let us denote  $f(M_0)h(M_0)^{-1}$  by  $c_0$ . Then  $f$  and  $c_0h$  are two regular functions on  $C$  which are identical on  $\mathrm{GL}(\mathbf{d})M_0$ . Since this orbit is dense in  $C$ , we deduce that  $f$  and  $c_0h$  are equal on  $C$ . The proof now follows.  $\square$

**2.4. The first fundamental theorem (FFT) for semi-invariants of bound quiver algebras.** Here, we first recall the construction of the so called generalized Schofield semi-invariants on module varieties, and then state the FFT for such semi-invariants. This is a fundamental result due to Derksen and Weyman [DW02], and Domokos [Dom02].

Let  $X$  be an  $A$ -module and let  $P_1 \xrightarrow{f} P_0 \rightarrow X \rightarrow 0$  be a fixed minimal projective presentation of  $X$  in  $\mathrm{mod}(A)$ . Let  $\theta^X \in \mathbb{Z}^{Q_0}$  be the integral weight defined so that  $\theta^X(v)$  equals the multiplicity of  $P_v$  in  $P_0$  minus the multiplicity of  $P_v$  in  $P_1$  for all  $v \in Q_0$ . Consequently, for an arbitrary  $A$ -module  $M$ , we have:

$$\begin{aligned} \theta^X(\mathbf{dim} M) &= \dim_K \mathrm{Hom}_A(P_0, M) - \dim_K \mathrm{Hom}_A(P_1, M) \\ &= \dim_K \mathrm{Hom}_A(X, M) - \dim_K \mathrm{Hom}_A(M, \tau X). \end{aligned}$$

Recall that the Auslander-Reiten translation of  $X$  is defined as  $\tau X = D(\mathrm{Coker} f^t)$  where  $(\_)^t = \mathrm{Hom}_A(\_, A)$  and  $D = \mathrm{Hom}_K(\_, K)$  (for more details, see [ASS06, §IV.2]). Let us point out that if the projective dimension of  $X$  is at most one then  $\theta^X(\mathbf{dim} M) = \dim_K \mathrm{Hom}_A(X, M) - \dim_K \mathrm{Ext}_A^1(X, M)$ .

Let  $\mathbf{d} \in \mathbb{Z}^{Q_0}$  be a dimension vector of  $A$  such that  $\theta^X(\mathbf{d}) = 0$ . In what follows, we explain how to construct the determinantal semi-invariant  $\bar{c}^X$  on  $\mathrm{mod}(A, \mathbf{d})$  of weight  $\theta^X$ .

First, note that for any module  $M \in \mathrm{mod}(A, \mathbf{d})$ ,  $\dim_K \mathrm{Hom}_A(P_0, M) = \dim_K \mathrm{Hom}_A(P_1, M)$  and hence the linear map

$$\mathrm{Hom}_A(f, M) : \mathrm{Hom}_A(P_0, M) \rightarrow \mathrm{Hom}_A(P_1, M)$$



can be viewed a square matrix. To write down this linear map in a more explicit form, let  $I$  and  $J$  be two index sets and let  $i \rightarrow \bar{i}$  and  $j \rightarrow \bar{j}$  be two functions  $I \rightarrow Q_0$  and  $J \rightarrow Q_0$  such that

$$P_1 = \bigoplus_{i \in I} P_{\bar{i}} \text{ and } P_0 = \bigoplus_{j \in J} P_{\bar{j}}.$$

Note that  $|\{i \in I \mid \bar{i} = x\}|$  is precisely the multiplicity of  $P_x$  in  $P_1$ , and similarly  $|\{j \in J \mid \bar{j} = x\}|$  is precisely the multiplicity of  $P_x$  in  $P_0$ . Moreover, we can view  $f$  as a matrix  $(f_{ji})_{j \in J, i \in I}$  where  $f_{ji}$  is a linear combination of paths from  $\bar{j}$  to  $\bar{i}$  in  $A$ . For each pair  $(j, i)$ , we fix a decomposition  $f_{ji} = (\sum c_{ji} p_{ji}) \bmod I$  where the  $p_{ji}$ 's are paths from  $\bar{j}$  to  $\bar{i}$  in  $KQ$  and the  $c_{ji}$ 's are non-zero scalars in  $K$ .

Writing  $\text{Hom}_A(P_1, M) = \bigoplus_{i \in I} M(\bar{i})$  and  $\text{Hom}_A(P_0, M) = \bigoplus_{j \in J} M(\bar{j})$ , we can view  $\text{Hom}_A(f, M)$  as the  $|I| \times |J|$  block matrix whose  $(i, j)$ -block-entry is the matrix  $\sum c_{ji} M(p_{ji}) \in \text{Mat}_{\mathbf{d}(\bar{i}) \times \mathbf{d}(\bar{j})}(K)$ ; denote this square matrix by  $d^X(M)$ . Now, we are ready to define:

$$\bar{\tau}^X : \text{mod}(A, \mathbf{d}) \rightarrow K, \quad \bar{\tau}^X(M) = \det d^X(M).$$

It is easy to see that  $\bar{\tau}^X$  is a semi-invariant of weight  $\theta^X$ . Moreover, any other choice of a minimal projective presentation of  $X$  leads to the same semi-invariant, up to a non-zero scalar. We call  $\bar{\tau}^X$  a *generalized Schofield semi-invariant*.

In what follows, an irreducible component  $C \subseteq \text{mod}(A, \mathbf{d})$  is said to be *faithful* if the intersection of the annihilators of all  $A$ -modules in  $C$  is zero.

**Theorem 9 (The FFT for semi-invariants).** *Let  $\mathbf{d}$  be a dimension vector of  $A$ ,  $C \subseteq \text{mod}(A, \mathbf{d})$  an irreducible component, and  $\theta \in \mathbb{Z}^{Q_0}$  an integral weight such that  $\theta(\mathbf{d}) = 0$ .*

- (1) ([DW02, Proposition 1] and [Dom02, Theorem 3.2]) *The weight space  $\text{SI}(C)_\theta$  is spanned by semi-invariants of the form  $\bar{\tau}^X$  such that  $\theta^X = \theta$ .*
- (2) ([DW02, Theorem 1]) *Assume that  $C$  is faithful. If  $\bar{\tau}^X$  is a non-zero semi-invariant in  $\text{SI}(C)$  then  $X$  has projective dimension at most one.*

As a direct consequence of the FFT, we have:

**Lemma 10.** *Let  $\mathbf{d}$  be a dimension vector of  $A$ , let  $C \subseteq \text{mod}(A, \mathbf{d})$  be a faithful irreducible component, and  $\theta \in \mathbb{Z}^{Q_0}$  an integral weight. Let*

$$C = \overline{C_1 \oplus \dots \oplus C_t}$$

*be the generic decomposition of  $C$  with  $C_i \subseteq \text{mod}(A, \mathbf{d}_i)$ ,  $1 \leq i \leq t$ , indecomposable irreducible components.*

*If  $\dim_K \text{SI}(C_i)_\theta \leq 1$  for all  $1 \leq i \leq t$  then  $\dim_K \text{SI}(C)_\theta \leq 1$ .*

*Proof.* Let  $\bar{\tau}^X \in \text{SI}(C)_\theta$  be a non-zero semi-invariant; in particular, we know that  $\theta^X(\mathbf{d}) = 0$ . Moreover, the projective dimension of  $X$  is at most one by FFT(2). It is now easy to see that  $\theta^X(\mathbf{d}_i) = 0$  for all  $1 \leq i \leq t$ .

Note also that for any  $(M_1, \dots, M_t) \in C_1 \times \dots \times C_t$ ,  $d^X(M_1 \oplus \dots \oplus M_t)$  is (equivalent to) the block diagonal matrix whose block diagonal entries are the square matrices  $d^X(M_1), \dots, d^X(M_t)$ . Consequently,  $\bar{\tau}^X(\bigoplus_{i=1}^t M_i) = \prod_{i=1}^t \bar{\tau}^X(M_i)$ . From this, we deduce that any two non-zero generalized Schofield semi-invariants in  $\text{SI}(C)_\theta$  are proportional, and so  $\dim \text{SI}(C)_\theta \leq 1$  by FFT(1).  $\square$

**2.5. Moduli spaces of modules.** Let  $\mathbf{d} \in \mathbb{Z}_{\geq 0}^{Q_0}$  be a dimension vector of  $A = KQ/I$  and let  $\theta \in \mathbb{Z}^{Q_0}$  be an integral weight of  $A$ .

Following King, an  $A$ -module  $M$  is said to be  $\theta$ -semi-stable if  $\theta(\dim M) = 0$  and  $\theta(\dim M') \leq 0$  for all submodules  $M' \leq M$ . We say that  $M$  is  $\theta$ -stable if  $M$  is non-zero,  $\theta(\dim M) = 0$ , and  $\theta(\dim M') < 0$  for all submodules  $\{0\} \neq M' < M$ . Now, consider the (possibly empty) open subsets

$$\text{mod}(A, \mathbf{d})_\theta^{ss} = \{M \in \text{mod}(A, \mathbf{d}) \mid M \text{ is } \theta\text{-semi-stable}\}$$

and

$$\text{mod}(A, \mathbf{d})_\theta^s = \{M \in \text{mod}(A, \mathbf{d}) \mid M \text{ is } \theta\text{-stable}\}$$

of  $\mathbf{d}$ -dimensional  $\theta$ (-semi)-stable  $A$ -modules.

Using methods from Geometric Invariant Theory, King showed in [Kin94] that the projective variety

$$\mathcal{M}(A, \mathbf{d})_\theta^{ss} := \text{Proj}\left(\bigoplus_{n \geq 0} \text{SI}(A, \mathbf{d})_{n\theta}\right)$$

is a GIT-quotient of  $\text{mod}(A, \mathbf{d})_\theta^{ss}$  by the action of  $\text{PGL}(\mathbf{d})$ . Here,  $\text{PGL}(\mathbf{d}) = \text{GL}(\mathbf{d})/T_1$  where  $T_1 = \{(\lambda \text{Id}_{\mathbf{d}(i)})_{i \in Q_0} \mid \lambda \in K^*\} \leq \text{GL}(\mathbf{d})$ . Note that there is a well-defined action of  $\text{PGL}(\mathbf{d})$  on  $\text{mod}(A, \mathbf{d})$  since  $T_1$  acts trivially on  $\text{mod}(A, \mathbf{d})$ . We say that  $\mathbf{d}$  is a  $\theta$ -semi-stable dimension vector if  $\text{mod}(A, \mathbf{d})_\theta^{ss} \neq \emptyset$ .

It was proved in [Kin94, Proposition 4.2] that the (closed) points of  $\mathcal{M}(A, \mathbf{d})_\theta^{ss}$  are in one-to-one correspondence with the isomorphism classes of those modules in  $\text{mod}(A, \mathbf{d})_\theta^{ss}$  that can be written as direct sums of  $\theta$ -stable modules. We call such  $A$ -modules  $\theta$ -polystable.

For an irreducible component  $C \subseteq \text{mod}(A, \mathbf{d})$ , we similarly define  $C_\theta^{ss}, C_\theta^s$ , and  $\mathcal{M}(C)_\theta^{ss}$ . One then has that the points of  $\mathcal{M}(C)_\theta^{ss}$  are in one-to-one correspondence with the isomorphism classes of  $\theta$ -polystable modules in  $C$ .

In general, it is difficult to describe/construct stable modules. The simple lemma below, which will be used in proving Theorem 3(2) and Proposition 33, identifies modules  $M$  which are  $\theta^M$ -stable. Recall that an  $A$ -module  $M$  is said to be *homogeneous* if  $M \simeq \tau M$ .

**Lemma 11.** *If  $M$  is a homogeneous Schur  $A$ -module then  $M$  is  $\theta^M$ -stable.*

*Proof.* Since  $M$  is homogeneous, we have that  $\theta^M(\dim M) = 0$ . Next, using that  $M$  is also Schur, we have that for any proper  $A$ -submodule  $0 \neq M' \subset M$ ,  $\text{Hom}_A(M, M') = 0$  and  $\dim_K \text{Hom}_A(M', \tau M) = \dim_K \text{Hom}_A(M', M) > 0$ . So,  $\theta^M(\dim M') < 0$  for all proper submodules  $M'$  of  $M$ .  $\square$

*Remark 12.* We should point out that in general the property that a module is Schur does not guarantee the existence of a weight with respect to which the module becomes (semi-)stable.

**2.6. DO and MF algebras with a preprojective component.** First, let us prove the following simple reduction result:

**Lemma 13.** *Any quotient of a MF bound quiver algebra is MF.*

*Proof.* Let  $A$  be a MF algebra,  $I$  an ideal of  $A$ , and  $\mathbf{d}$  a dimension vector of  $A$ . Then, any irreducible component  $C \subseteq \text{mod}(A/I, \mathbf{d})$  is embedded ( $\text{GL}(\mathbf{d})$ -equivariantly) in an irreducible component  $C' \subseteq \text{mod}(A, \mathbf{d})$ . Since the base field is assumed to be of characteristic zero, we know from invariant theory [DK02, Corollary 2.2.9] that the above embedding gives rise to a surjective map at the level of  $\text{SL}(\mathbf{d})$ -invariant rings; in particular, for any character  $\chi \in X^*(\text{GL}(\mathbf{d}))$ ,  $\dim_K \text{SI}(C)_\chi \leq \dim_K \text{SI}(C')_\chi$ . The proof now follows.  $\square$



Now, we are ready to prove Theorem 1:

*Proof of Theorem 1.* We have seen that the implications  $(1) \implies (2) \implies (3)$  hold true for arbitrary bound quiver algebras.

It remains to prove the implication  $(3) \implies (1)$ . Assume to the contrary that the MF algebra  $A$  is representation-infinite. It essentially follows from the work of Happel and Vossieck that any connected algebra admitting a preprojective component has a tame concealed algebra as a quotient (see [SS07a, Theorem XIV.3.1]). This result combined with Lemma 13 tells us that  $A$  has a quotient  $B$  which is a MF tame concealed algebra. Denote by  $\mathbf{h}$  the dimension vector of an indecomposable  $B$ -module lying at the mouth of a homogeneous tube of  $B$ . It is well-known that  $\text{mod}(B, \mathbf{h})$  is irreducible and that there is always an integral weight  $\theta$  of  $B$  such that  $\mathcal{M}(B, \mathbf{h})_{\theta}^{ss} \simeq \mathbb{P}^1$  (see for example [Chi11]). In particular,  $\text{SI}(B, \mathbf{h})$  is not multiplicity-free (contradiction).  $\square$

### 3. REPRESENTATION-INFINITE DO ALGEBRAS

Our example of a representation-infinite DO algebra works in arbitrary characteristic and is given by the following quiver with relations:

$$(1) \quad \begin{array}{ccc} \sigma \curvearrowright \bullet & \xrightarrow{\nu} & \bullet \curvearrowright \rho \\ 1 & & 2 \end{array} \quad \sigma^4 = \rho^4 = \rho^2\nu = 0, \quad \nu\sigma = \rho\nu$$

It is minimal representation-infinite with distributive ideal lattice (see, for example [BGRS85, p. 242]) and also known to be tame ([Sko86] or [HM88]). We denote this algebra by  $A$  throughout this section.

The proof of Theorem 2, which states that every irreducible component of every module variety of  $A$  has a dense orbit, proceeds in the following steps. Let  $(l, m)$  be a dimension vector for  $A$ , and fix an irreducible component  $C$  of  $\text{mod}(A, (l, m))$ . On an open set of  $C$ , the Jordan types of the endomorphisms associated to  $\sigma, \rho$  will be constant so we may fix these and work with their isotropy groups acting on a subvariety of  $C$ . Explicit matrix calculations allow us to reduce the problem to working with matrices with few nonzero entries, at the cost of complicating the remaining group acting on these matrices. After sufficient reduction we are left with a classical matrix problem, which can be stated in the language of representations of posets, as introduced by Nazarova and Roiter [Naz81]. A classification theorem of M. Kleiner finally allows us to conclude that the component  $C$  has a dense orbit.

By an abuse of notation, we will also use  $\sigma, \nu, \rho \in A$  to denote the matrices they act by in various representations (i.e., at various points of  $\text{mod}(A, (l, m))$ ). Let  $G = \text{GL}(l) \times \text{GL}(m)$ , and recall that the action of  $G$  on  $\text{mod}(A, (l, m))$  is given by

$$(2) \quad (g_1, g_2) \cdot (\sigma, \nu, \rho) = (g_1\sigma g_1^{-1}, g_2\nu g_1^{-1}, g_2\rho g_2^{-1}).$$

**3.1. Initial reductions.** To show that each component  $C$  of  $\text{mod}(A, (l, m))$  has a dense orbit, we may assume that a general representation of  $C$  is indecomposable, by Remark 5. There is an open subset of  $C$  on which the Jordan types of  $\sigma$  and  $\rho$  are constant, by semicontinuity of  $\dim_K \text{Im } \sigma^i$  and  $\dim_K \ker \rho^i$ . In this case we say that  $C$  is of type  $(\lambda, \mu)$ , where  $\lambda = (\lambda_1, \lambda_2, \dots)$  and  $\mu = (\mu_1, \mu_2, \dots)$  are partitions whose conjugates list the sizes of the Jordan blocks of  $\sigma$  and  $\rho$ , respectively. (We use this convention of associating a partition to a nilpotent endomorphism throughout.)

Thus we may use the action of  $G$  to fix representatives  $\sigma_0, \rho_0$  in the following block form, where  $I_k$  denotes a  $k \times k$  identity matrix, and vertical and horizontal bars indicate further subdivision of some blocks:

$$(3) \quad \sigma_0 = \begin{pmatrix} 0 & & & \\ I_{\lambda_2}|0 & 0 & & \\ 0 & I_{\lambda_3}|0 & 0 & \\ 0 & 0 & I_{\lambda_4}|0 & 0 \end{pmatrix} \quad \rho_0 = \begin{pmatrix} 0 & \frac{I_{\mu_2}}{0} & 0 & 0 \\ 0 & \frac{I_{\mu_3}}{0} & 0 & \\ & 0 & \frac{I_{\mu_4}}{0} & \\ & & 0 & 0 \end{pmatrix}$$

Let  $I(\sigma_0) \subset \text{GL}(l)$ ,  $I(\rho_0) \subset \text{GL}(m)$  be the stabilizers of  $\sigma_0, \rho_0$ , respectively, and set  $H := I(\sigma_0) \times I(\rho_0)$ . We are reduced to showing that  $H$  has a dense orbit on

$$N = \{\nu \in \text{Hom}_K(K^l, K^m) \mid \rho_0 \nu = \nu \sigma_0\},$$

noting that the relation  $\rho_0 \nu = \nu \sigma_0$  gives that  $\nu$  is of the form

$$\nu = \begin{pmatrix} \nu_{11} & \nu_{12} & 0 & 0 \\ \nu_{21} & 0 & & \\ 0 & & & \\ 0 & & & \end{pmatrix}$$

with

$$\nu_{12} = \begin{pmatrix} M \\ 0 \end{pmatrix} \quad \nu_{21} = (M \quad 0)$$

where  $M$  is a generic  $\mu_2 \times \lambda_2$  matrix.

**3.2. Action of the isotropy group.** The isotropy group  $H$  will be described in terms of row and column operations, and then we will see how these act on  $\nu$ . For  $i, j \in \mathbb{N}$  and  $\alpha \in K$ , let  $E_{ij}^\alpha$  be the elementary matrix that adds  $\alpha$  times row  $i$  to row  $j$  when acting by left multiplication; by a slight abuse of notation, we use this for elements of both  $P_\lambda$  and  $P_\mu$ . Say that  $E_{ij}^\alpha$  acts *upwards* when  $j \leq i$ , and *downwards* when  $i \leq j$ . Similarly,  $(E_{ij}^\alpha)^{-1}$  acts on the right of a matrix by subtracting  $\alpha$  times column  $j$  from column  $i$ ; say it acts *leftwards* when  $i \leq j$  and *rightwards* when  $j \leq i$ .

The following technical lemma describes the stabilizers  $I(\sigma_0)$  and  $I(\rho_0)$ , and can be thought of as giving a list of “admissible” column operations on  $\nu$  by  $I(\sigma_0)$  and row operations by  $I(\rho_0)$ . While it initially looks quite messy, the effect of these operations on  $\nu$  is dramatically simplified by the fact that  $\nu$  has only three nonzero blocks, and two of these are essentially the same. So while the elements of the stabilizer generally perform several simultaneous row and column operations on  $\nu$ , we really only need to visualize one of these operations at a time.

**Lemma 14.** *The isotropy group  $I(\sigma_0)$  contains  $E_{ij}^\alpha$  for  $(i, j)$  in the following sets:*

$$(4) \quad \{1 \leq i \leq \lambda_1, \lambda_2 < j \leq \lambda_1\}, \quad \{1 \leq i \leq \lambda_1, \lambda_1 + \lambda_3 < j \leq \lambda_1 + \lambda_2\};$$

*it contains  $E_{\lambda_1+i \lambda_1+j}^\alpha E_{ij}^\alpha$  for  $(i, j)$  in*

$$(5) \quad \{1 \leq i \leq \lambda_2, \lambda_3 < j \leq \lambda_2\}, \quad \{1 \leq i \leq \lambda_2, \lambda_1 + \lambda_4 < j \leq \lambda_1 + \lambda_3\};$$

*it contains  $E_{\lambda_1+\lambda_2+i \lambda_1+\lambda_2+j}^\alpha E_{\lambda_1+i \lambda_1+j}^\alpha E_{ij}^\alpha$  for  $(i, j)$  in*

$$(6) \quad \{1 \leq i \leq \lambda_3, \lambda_4 < j \leq \lambda_3\}, \quad \{1 \leq i \leq \lambda_3, \lambda_1 < j \leq \lambda_1 + \lambda_4\};$$

and it contains  $E_{\lambda_1+\lambda_2+\lambda_3+i \ \lambda_1+\lambda_2+\lambda_3+j}^\alpha E_{\lambda_1+\lambda_2+i \ \lambda_1+\lambda_2+j}^\alpha E_{\lambda_1+i \ \lambda_1+j}^\alpha$  for  $(i, j)$  in

$$(7) \quad \{1 \leq i \leq \lambda_4, \ 0 < j \leq \lambda_4\}.$$

Element of  $I(\rho_0)$  can be obtained by replacing  $\lambda$  with  $\mu$  and reversing the role of  $i, j$  in the above sets.

*Proof.* Since  $\text{GL}(l)$  acts by simultaneous row and column operations on  $\text{End}_K(K^l)$ , the lemma follows by taking each subset and examining its effect on  $\sigma_0$ . Given the elements of  $I(\sigma_0)$ , we note that  $\rho_0$  is just the transpose of  $\sigma_0$  after replacing  $\lambda$  with  $\mu$ , so this gives the statement about  $I(\rho_0)$ .  $\square$

First we understand the action of  $H$  on  $\nu_{12}$  and  $\nu_{21}$ . Since  $H$  preserves the relation  $\rho_0\nu = \nu\sigma_0$ , it acts simultaneously in the same way on the  $M$  in  $\nu_{12}$  and  $\nu_{21}$ ; consequently we can unambiguously speak of the action of  $H$  on  $M$ . We ignore the action on  $\nu_{11}$  for the time being, since we deal with a general  $\nu$ . For column operations, every  $E_{ij}^\alpha$  with  $i \leq j$  appears as a factor of some element of  $I(\sigma_0)$  in Lemma 14, so we at least have column operations leftward in  $M$ . Similarly, we have every  $E_{ij}^\alpha$  with  $\lambda_1 \geq i \geq j \geq 1$  appearing as a factor of some element of  $I(\rho_0)$ , so we have row operations upward. The other factors of these elements of  $H$  act trivially on  $\nu$  by adding rows or columns of zeros to other rows or columns.

Assume for the rest of this section that  $\mu_2 \geq \lambda_2$ , since the case where the inequality is reversed follows by a similar argument. The admissible row and column operations of the last paragraph allow us to take  $M = \begin{pmatrix} 0 \\ I_{\lambda_2} \end{pmatrix}$  from now on, which leaves the upper-left block  $2 \times 2$  part of  $\nu$  to be

$$(8) \quad \left( \begin{array}{c|c} \nu_{11} & \nu_{12} \\ \hline \nu_{21} & 0 \end{array} \right) = \begin{array}{c} \lambda_2 \quad \lambda_1 - \lambda_2 \mid \lambda_2 \\ \mu_2 - \lambda_2 \left( \begin{array}{cc|c} * & * & 0 \\ * & * & I_{\lambda_2} \\ * & * & 0 \end{array} \right) \\ \mu_1 - \mu_2 \\ \mu_2 - \lambda_2 \left( \begin{array}{cc|c} 0 & 0 & 0 \\ I_{\lambda_2} & 0 & 0 \end{array} \right) \\ \lambda_2 \end{array}$$

where the labels of the rows and columns indicate the sizes of a further subdivision of this upper-left quadrant of  $\nu$ .

Now we notice that for  $C$  to be an indecomposable component, we must have either  $\lambda_1 = \lambda_2$  or  $\mu_1 = \mu_2$ , because if both were nonzero we could get a 1 in the lower right corner of  $\nu_{11}$  and then use elements of  $H$  to clear the entire column above it and row to its left, giving a direct summand with  $V_1 = V_2 = K$ ,  $\sigma = \rho = 0$ , and  $\nu = \text{Id}$ . We will use this fact further below but keep the two cases together for now.

Now we examine operations that  $H$  can do on  $\nu_{11}$  from rows of  $\nu_{21}$  and columns of  $\nu_{12}$ . Lemma 14 allows us to add multiples of column  $\lambda_1 + j$  to column  $j_0$  for any  $j_0 \leq j \leq \lambda_2$ . Since the only nonzero part of column  $\lambda_1 + j$  is in  $\nu_{12}$ , this will only change  $\nu_{11}$ . Similarly we may add multiples of row  $\mu_1 + i$  to row  $i_0$  for  $i_0 \leq i \leq \mu_2$  and affect only  $\nu_{11}$ . (Again, in each case these are followed by row or column operations from higher blocks of  $\nu$ , but those rows and columns are zero.)

We show that such operations allow us to use the action of  $H$  to put  $\nu$  of the form

$$(9) \quad \left( \begin{array}{c|c} \nu_{11} & \nu_{12} \\ \hline \nu_{21} & 0 \end{array} \right) = \frac{\begin{array}{c} \mu_2 - \lambda_2 \\ \lambda_2 \\ \mu_1 - \mu_2 \\ \mu_2 - \lambda_2 \\ \lambda_2 \end{array}}{\begin{array}{c} \lambda_2 \quad \lambda_1 - \lambda_2 \end{array}} \left( \begin{array}{cc|c} 0 & * & 0 \\ 0 & * & I_{\lambda_2} \\ * & * & 0 \\ 0 & 0 & 0 \\ I_{\lambda_2} & 0 & 0 \end{array} \right).$$

Given a pair  $(i_0, j_0)$  with  $1 \leq i_0 \leq \mu_2$  and  $1 \leq j_0 \leq \lambda_2$ , we need to show that one of the following holds in order to clear that entry of  $\nu_{11}$ :

- (i) In (8) there is a 1 in row  $i_0$  and some column  $j \geq \lambda_1 + j_0$ ; or
- (ii) In (8) there is a 1 in column  $j_0$  and some row  $i \geq \mu_1 + i_0$ .

Suppose that (ii) does not hold, so the 1 in column  $j_0$  of  $\nu_{21}$  is in some row less than  $\mu_1 + i_0$ . Now examining (8), we see that if we write  $j_0 = \lambda_2 - m$  (so  $0 \leq m < \lambda_2$  counts columns backwards from the right of the lower  $I_{\lambda_2}$ ), this 1 is precisely in row  $\mu_1 + \mu_2 - m$ . This gives

$$\mu_1 + \mu_2 - m < \mu_1 + i_0$$

and so  $i_0 > \mu_2 - m$ . Now let us see that (i) holds. First, there is a 1 in row  $i_0$  and some column of  $\nu_{12}$  because  $m < \lambda_2$ , so  $i_0 > \mu_2 - \lambda_2$  which we can see in (8) has a 1 in that row. We want to see what column this 1 is in. Write  $i_0 = \mu_2 - n$ , so that by examining the upper-right block of (8) we see that the 1 is in column  $\lambda_1 + \lambda_2 - n = \lambda_1 + (j_0 + m) - n$ . But we also have that  $i_0 = \mu_2 - n > \mu_2 - m$ , so that  $m > n$ . This shows that  $\lambda_1 + j_0 + (m - n) > \lambda_1 + j_0$ , so that (i) is satisfied. Thus we can use the action of  $H$  to put  $\nu$  in the form (9).

**3.3. Conversion to a matrix problem.** To finally show that  $H$  has a dense orbit on  $N$ , we *a priori* break the problem into two cases, first considering when  $\mu_1 = \mu_2$ ; but we will see later that the other case already follows from this one.

Now we have the upper left  $2 \times 2$  block part of  $\nu$  looks like

$$(10) \quad \left( \begin{array}{c|c} \nu_{11} & \nu_{12} \\ \hline \nu_{21} & 0 \end{array} \right) = \frac{\begin{array}{c} \mu_2 - \lambda_2 \\ \lambda_2 \\ \mu_2 - \lambda_2 \\ \lambda_2 \end{array}}{\begin{array}{c} \lambda_2 \quad \lambda_1 - \lambda_2 \end{array}} \left( \begin{array}{cc|c} 0 & M_1 & 0 \\ 0 & M_2 & I_{\lambda_2} \\ 0 & 0 & 0 \\ I_{\lambda_2} & 0 & 0 \end{array} \right)$$

with  $M_1, M_2$  general matrices. We now consider the subgroup  $H' \subset H$  which acts on matrices of the form (10) and see that  $H'$  has a dense orbit on this space. This matrix problem may appear to be quite complicated because we don't have a nice description of  $H'$ ; it requires case-by-case analysis depending on the relative sizes of the parts of  $\lambda, \mu$ . The key is that it can be viewed as a classical problem on linear representations of posets.

We have that  $\lambda_2 = \lambda_3$  since we assumed that  $C$  is an indecomposable component; otherwise, we would be able to use  $H'$  to make operations from the far right column of  $\nu_{12}$  to clear the bottom row of  $M_2$ , splitting off a direct summand of dimension  $(1, 0)$  from a general representation.

**Lemma 15.** *The group  $H'$  includes the following operations on  $\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$ :*

- (1) *Arbitrary column operations*

(2) Row operations downwards within blocks of size

$$(11) \quad \begin{pmatrix} \mu_4 \\ \dots\dots\dots \\ \mu_3 - \mu_4 \\ \dots\dots\dots \\ \mu_2 - \mu_3 \end{pmatrix}$$

(3) Row operations upwards within blocks of size

$$(12) \quad \begin{pmatrix} \mu_2 - \lambda_2 \\ - - - - - \\ \lambda_4 \\ - - - - - \\ \lambda_3 - \lambda_4 \end{pmatrix}$$

*Proof.* We get (1) immediately from the first set of (4) in Lemma 14; since there are only 0s below  $\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$  the form (10) is already preserved.

For (2), Lemma 14 allows us to make downward row operations in  $\nu$  within the stated blocks while at least stabilizing  $\sigma_0$  and  $\rho_0$ , say from row  $i$  to row  $j > i$ . If  $i \leq \mu_2 - \lambda_2$ , we are done. But if  $i > \mu_2 - \lambda_2$ , this will cause a nonzero entry in row  $j$  column  $\lambda_1 + i - (\mu_2 - \lambda_2)$ , which we must remove. Fortunately, there is a 1 to the right of this in row  $j$  column  $\lambda_1 + j - (\mu_2 - \lambda_2)$ , with the rest of that column zero, so Lemma 14 allows a column operation leftward to clear our undesirable entry, with the net effect being just a row operation in  $\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$  leaving the rest of  $\nu$  unchanged.

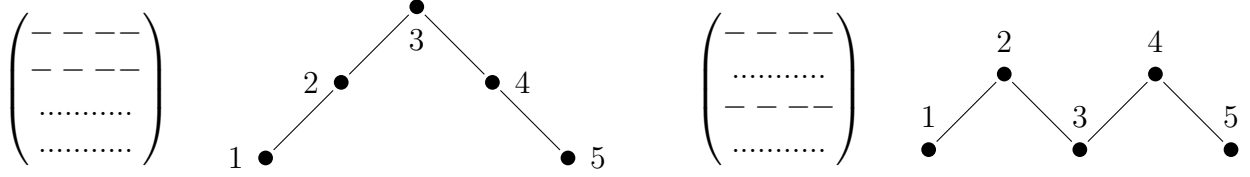
For (3), we initially may make any row operations upwards in these blocks while stabilizing  $\sigma_0, \rho_0$ , say from row  $i$  to row  $j < i$ . If  $i \leq \mu_2 - \lambda_2$ , the row consists of zeros outside of  $\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$  so we are done. But if  $i > \mu_2 - \lambda_2$ , this causes a nonzero entry in row  $j$  column  $\lambda_1 + i - (\mu_2 - \lambda_2)$  which is to the right of the 1 in its row in  $\nu_{12}$  (note that  $j > \mu_2 - \lambda_2$  also in this case, by the block form). From Lemma 14 (6) (7), we are only allowed a rightward column operation from the 1 in row  $j$  column  $\lambda_1 + j - (\mu_2 - \lambda_2)$  to the newly created nonzero entry if  $i - (\mu_2 - \lambda_2)$  and  $j - (\mu_2 - \lambda_2)$  are both either less than or equal to  $\lambda_4$ , or both greater than  $\lambda_4$  but less than or equal to  $\lambda_3$ . This is precisely the condition illustrated in (12), so we are done.  $\square$

By putting the two dashed and two dotted lines in any order (depending on  $\lambda, \mu$ ) we get 5 blocks total, giving a classical matrix problem: we have a block form matrix in which we are allowed all column operations, all row operations within blocks, and row operations from some blocks to others. Such a problem is equivalent to studying the linear representations of a certain partially ordered set, whose elements are the blocks  $\{B_i\}$ , with the relations  $B_i \preceq B_j$  whenever adding rows of  $B_i$  to  $B_j$  is admissible in the given matrix problem [Sim92, Ch. 2.1].

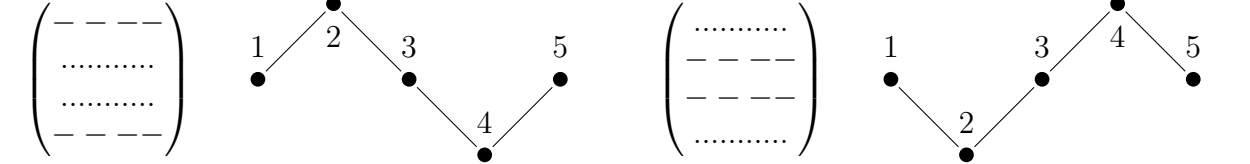
Returning to our problem, there are six possibilities occurring, which are all possible combinations of interspersing the dotted and dashed lines. We give the Hasse diagram of the corresponding poset, with the vertex labels corresponding to labeling the blocks 1 through

5, starting at the top.

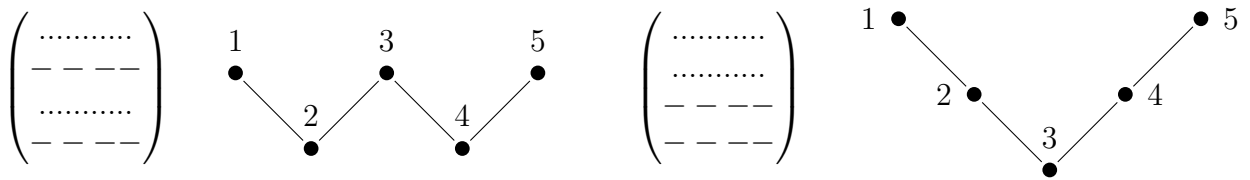
(13)



(14)



(15)



In general, the prescribed matrix operations act with finitely many orbits on the matrix space precisely when the corresponding partially ordered set admits only finite many representations of a given dimension; by M. Kleiner's classification [Kle72], the posets obtained above are all of finite type (see also [Sim92, Theorem 10.1]). So we are done if  $\mu_1 = \mu_2$ .

This leaves only the case that  $\lambda_1 = \lambda_2$ , where  $\nu$  is of the form

$$(16) \quad \left( \begin{array}{c|c} \nu_{11} & \nu_{12} \\ \nu_{21} & 0 \end{array} \right) = \frac{\begin{array}{c} \mu_2 - \lambda_2 \\ \lambda_2 \\ \mu_1 - \mu_2 \\ \mu_2 - \lambda_2 \\ \lambda_2 \end{array}}{\begin{array}{c} \lambda_2 \\ \lambda_2 \\ M_3 \\ 0 \\ I_{\lambda_2} \end{array}} \left( \begin{array}{c|c} \lambda_2 & \lambda_2 \\ 0 & 0 \\ 0 & I_{\lambda_2} \\ 0 & 0 \\ 0 & 0 \end{array} \right).$$

The initial reduction that we may take  $\mu_2 = \mu_3$  is the transpose of an argument above: otherwise, there would be a row operation from the bottom row of  $\nu_{21}$  to any of the first  $\mu_1$  rows of  $\nu$ , allowing us to take the 1 in the bottom right of  $\nu_{21}$  and clear the column above it. This would produce a direct summand of dimension  $(2, 0)$  in a general representation of  $C$ , contradicting the indecomposability of this component.

At this point we are left with the transposition of the matrix problem above on  $M_3$ , except that it is even slightly simpler because there is no zero block to the left of  $I_{\lambda_2}$  in  $\nu_{21}$ , so there will be one less dashed line when represented as above. Then it follows that the matrix space has a dense orbit as a special case of our previous analysis for  $\mu_1 = \mu_2$ , by taking  $\lambda_4 = 0$  there.

This completes the proof of Theorem 2.

*Remark 16.* Solving the corresponding matrix (or poset) problem even gives an explicit representation in the dense orbit of each component.



#### 4. REPRESENTATION-INFINITE MF ALGEBRAS

In this section, we first give examples of algebras which are triangular and representation-infinite but have the MF property (Theorem 18), and then prove Theorem 3(2). We also classify all MF string algebras.

**4.1. An example of an MF representation-infinite string algebra.** Let us first recall some basic notions on string algebras and their indecomposable modules.

A bound quiver algebra  $A = KQ/I$ , where  $I$  is generated by a finite set  $\mathcal{R}$  of relations, is said to be a *string algebra* if the following conditions are satisfied:

- (1) each vertex of  $Q$  is the tail of at most two arrows, and the head of at most two arrows;
- (2) each relation in  $\mathcal{R}$  is just a monomial in the arrows of  $Q$ ;
- (3) for each arrow  $b \in Q_1$ , there is at most one arrow  $a \in Q_1$  with  $ta = hb$  and at most one arrow  $c \in Q_1$  with  $tb = hc$  such that  $ab \notin \mathcal{R}$  and  $bc \notin \mathcal{R}$ .

The indecomposable modules over string algebras can be described using certain paths consisting of arrows and their inverses. For each arrow  $a \in Q_1$ , we denote by  $a^{-1}$  its formal inverse for which we define  $ta^{-1} = ha$  and  $ha^{-1} = ta$ . We also define  $(a^{-1})^{-1}$  to be precisely  $a$ . Denote the set of formal inverses of arrows by  $Q_1^{-1}$ . We define a *string* of length  $n \geq 1$  to be a word  $C = w_1 w_2 \dots w_n$  with  $w_1, \dots, w_n \in Q_1 \cup Q_1^{-1}$  such that: (i)  $tw_i = hw_{i+1}$ ,  $\forall 1 \leq i \leq n-1$ ; (ii)  $C$  does not contain any subword of the form  $aa^{-1}$  with  $a \in Q_1 \cup Q_1^{-1}$ ; (iii)  $C$  does not contain any subword  $C'$  with  $C'$  or  $(C')^{-1}$  in  $\mathcal{R}$ . Given a string  $C$ , we denote the corresponding *string  $A$ -module* by  $M(C)$ .

We define a *band* to be a string such that all of its positive powers are defined. A band  $B$  is said to be *primitive* if it is not periodic, i.e. there exists no string  $C$  such that  $B = C^n$  for some  $n \geq 2$ . Given a band  $B$ , an integer  $n \in \mathbb{Z}_{>0}$ , and scalars  $\lambda_1, \dots, \lambda_n \in K^*$ , we denote the corresponding *band  $A$ -module* by  $M(B, \lambda_1, \dots, \lambda_n)$ . We also denote  $M(B, \lambda, \dots, \lambda)$  by  $M(B, \lambda, n)$ ; when  $n = 1$ , we simply write  $M(B, \lambda)$ . For the details behind the construction of string and band modules, we refer the reader to [BR87, WW85].

We define  $\mathcal{S}$  to be the set of equivalence classes of strings where two strings  $C$  and  $C'$  are said to be equivalent if  $C'$  or its inverse equals  $C$ . We define  $\mathcal{B}$  to be the set of equivalence classes of primitive bands where two such bands  $B = w_1 \dots w_n$  and  $B'$  are said to be equivalent if  $B'$  or its inverse is either  $B$  or of the form  $w_i \dots w_n w_1 \dots w_{i-1}$  for some  $2 \leq i \leq n$ . Now we are ready to state the following fundamental facts about modules over string algebras:

- If  $M(B, \lambda_1, \dots, \lambda_n)$  is a band module with  $\lambda_i \neq \lambda_{i+1}$  for some  $1 \leq i \leq n-1$  then

$$M(B, \lambda_1, \dots, \lambda_n) \simeq M(B, \lambda_1, \dots, \lambda_i) \oplus M(B, \lambda_{i+1}, \dots, \lambda_n);$$

- the string modules  $M(C)$  with  $C \in \mathcal{S}$  and the band modules  $M(B, \lambda, n)$  with  $B \in \mathcal{B}$ ,  $\lambda \in K^*$ , and  $n \geq 1$ , from a complete set of representatives of the isomorphism classes of indecomposable  $A$ -modules.
- for any  $A$ -module  $M$ , we have

$$\sum_{a \in Q_1} \text{rank } M(a) = \dim_K M - s,$$

where  $s$  is the number of string indecomposable modules occurring in a direct sum decomposition of  $M$  into indecomposable modules.

Let  $\mathbf{d}$  be a dimension vector of the string algebra  $A = KQ/I$ . Let us consider a family  $\mathcal{F} = (C_1, \dots, C_l, (B_1, m_1), \dots, (B_n, m_n))$  consisting of strings  $C_1, \dots, C_l$ , primitive bands  $B_1, \dots, B_n$ , and positive integers  $m_1, \dots, m_n$  such that  $\mathbf{d} = \sum_{i=1}^l \mathbf{dim} M(C_i) + \sum_{j=1}^n \mathbf{dim} M(B_j, 1, m_j)$ . Now, consider the regular morphism:

$$f : \mathrm{GL}(\mathbf{d}) \times (K^*)^n \rightarrow \mathrm{mod}(A, \mathbf{d})$$

$$(g, \lambda_1, \dots, \lambda_n) \rightarrow g\left(\bigoplus_{i=1}^l M(C_i) \oplus \bigoplus_{j=1}^n M(B, \lambda_j, m_j)\right)$$

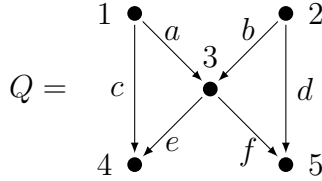
We denote  $\overline{\mathrm{Im} f}$  by the same letter  $\mathcal{F}$ . Note that the closed subvarieties of the form  $\mathcal{F}$  are irreducible, and  $\mathrm{mod}(A, \mathbf{d})$  is covered by finitely many of them. Consequently, the irreducible components of  $\mathrm{mod}(A, \mathbf{d})$  are among the  $\mathcal{F}$ 's.

It is now clear that if  $C \subseteq \mathrm{mod}(A, \mathbf{d})$  is an indecomposable irreducible component then either:

- $C = \overline{\mathrm{GL}(\mathbf{d})M(T)}$  for some string  $T$ ; or else
- $C = \overline{\bigcup_{\lambda \in K^*} \mathrm{GL}(\mathbf{d})M(B, \lambda, m)}$  for some primitive band  $B$  and positive integer  $m$ . In fact,  $m$  has to be 1 since otherwise  $C$  would have to be  $\overline{\bigcup \mathrm{GL}(\mathbf{d})M(B, \lambda_1, \dots, \lambda_m)}$  where the union is over all  $m$ -tuples  $(\lambda_1, \dots, \lambda_m) \in (K^*)^m$  with  $\lambda_i \neq \lambda_j, \forall 1 \leq i \neq j \leq m$ ; in particular, the generic module in  $C$  would be decomposable. In conclusion,

$$C = \overline{\bigcup_{\lambda \in K^*} \mathrm{GL}(\mathbf{d})M(B, \lambda)}.$$

For the remainder of this subsection, we will be working with the string algebra  $\Lambda = KQ/I$  where



and  $I$  is the ideal generated by  $ea, eb, fa, fb$ .

**Lemma 17.**  $\Lambda$  is a string algebra with

$$B_0 = c^{-1}ef^{-1}db^{-1}a$$

the only primitive band. The indecomposable band modules of  $\Lambda$  occur in dimension vectors  $m\mathbf{d}_0$  where  $\mathbf{d}_0 = (1, 1, 2, 1, 1)$  and  $m \geq 1$ . Moreover,  $\Lambda$  is a minimal representation-infinite algebra.

*Proof.* Let  $B$  be a band of  $\Lambda$ . It has to be supported on the whole quiver, because it cannot be supported only on the left or right triangle. Thus without loss of generality we can assume the band starts with  $a$  (from right to left). We are forced to continue with  $b^{-1}, d, f^{-1}$ , and then we are forced to continue with  $e, c^{-1}$ , etc. Since we have to finish at vertex 1, the band  $B$  needs to be a power of  $B_0$ .

The algebra  $\Lambda$  is clearly of infinite representation type with a  $K$ -basis consisting of the  $e_i$ ,  $1 \leq i \leq 5$ , and  $a, b, c, d, e, f$ . Hence, for any non-zero ideal  $J$  of  $\Lambda$  one of the  $e_i$ 's or

arrows would have to be in  $J$ , so  $\Lambda/J$  is of finite representation type, i.e.  $\Lambda$  is minimal representation-infinite.  $\square$

Now, we are ready to prove:

**Theorem 18.** *The algebra  $\Lambda$  is MF.*

*Proof.* Let  $C \subseteq \text{mod}(\Lambda, \mathbf{d})$  be an irreducible component. If  $C$  is not faithful then it is an irreducible component for a representation-finite algebra in which case  $\text{SI}(C)$  is clearly multiplicity-free.

Now, let us assume that  $C$  is faithful. In fact, according to Lemma 10, we can assume that  $C$  is indecomposable. If  $C$  is the orbit closure of a string module then  $\text{SI}(C)$  is again multiplicity-free. Otherwise, we know from the discussion above and Lemma 17 that

$$C = \overline{\bigcup_{\lambda \in K^*} \text{GL}(\mathbf{d}_0)M(B_0, \lambda)} \subseteq \text{mod}(\Lambda, \mathbf{d}_0).$$

Then, any  $M \in C$  satisfies the rank conditions

$$(17) \quad \text{rank}(M(a)|M(b)) \leq 1 \quad \text{and} \quad \text{rank}(M(e)^t|M(f)^t) \leq 1$$

since these conditions are satisfied by the  $\mathbf{d}_0$ -dimensional band modules; in particular,  $M$  has a submodule  $M'$  of dimension vector  $(1, 1, 1, 1, 1)$  and a submodule  $M''$  of dimension vector  $(0, 0, 1, 0, 0)$ .

Let  $C'$  be the closed subvariety of  $\text{mod}(\Lambda, \mathbf{d}_0)$  consisting of those modules that satisfy the two rank conditions above. We will show that  $\text{SI}(C')$  is multiplicity-free which in turn implies that  $\text{SI}(C)$  is multiplicity-free.

Now, let  $\theta \in \mathbb{Z}^{Q_0}$  be a weight such that  $\text{SI}(C')_\theta \neq 0$ ; in particular,  $\theta(\mathbf{d}_0) = 0$ . We claim that  $\theta(3) = 0$ . Indeed, let  $M \in C'$  be so that  $f(M) \neq 0$  for some  $f \in \text{SI}(C')_\theta$ . Then,  $M$  is  $\theta$ -semi-stable and so  $\theta(\dim M') \leq 0$  and  $\theta(\dim M'') \leq 0$ . It now follows that  $\theta(3) = 0$ .

Next, write  $K[C'] = K[\text{Mat}_{\mathbf{d}_0(4) \times \mathbf{d}_0(1)}(K) \times \text{Mat}_{\mathbf{d}_0(5) \times \mathbf{d}_0(2)}(K) \times C'']$  with  $C''$  the appropriate affine variety. Then, using the claim above, we have:

$$K[C']^{\text{SL}(\mathbf{d}_0)} = K[\text{Mat}_{\mathbf{d}_0(4) \times \mathbf{d}_0(1)}(K)] \otimes K[\text{Mat}_{\mathbf{d}_0(5) \times \mathbf{d}_0(2)}(K)] \otimes K[C'']^{\text{GL}(\mathbf{d}_0(3))}$$

But  $K[C'']^{\text{GL}(\mathbf{d}_0(3))} = K$  since this ring of invariants is generated by polynomials in the entries of  $M(ea), M(fa), M(eb), M(fb)$ . This can be easily seen directly or by simply quoting the FFT for  $\text{GL}(2, K)$ .

It is now clear that  $\text{SI}(C)^{\text{SL}(\mathbf{d}_0)}$  is multiplicity-free with each non-zero weight space  $\text{SI}(C)_\theta$  being spanned by  $\det_c^m \cdot \det_d^n$  where  $\theta = (m, n, 0, -m, -n)$ , and  $\det_c \in \text{SI}(C)_{(1,0,0,-1,0)}$ ,  $\det_d \in \text{SI}(C)_{(0,1,0,0,-1)}$  are the obvious determinant functions along the two specified arrows. This finishes the proof.  $\square$

*Remark 19.* We will see below that the algebra  $\Lambda$  is also Schur-representation-finite.

**4.2. The tame case.** Let  $A = KQ/I$  be a bound quiver algebra,  $\mathbf{d} \in \mathbb{Z}_{\geq 0}^{Q_0}$  a dimension vector of  $A$ ,  $C$  an irreducible component of  $\text{mod}(A, \mathbf{d})$ , and  $\theta \in \mathbb{Z}^{Q_0}$  an integral weight. Denote by  $C_\theta^{ss}$  the  $\theta$ -semi-stable locus in  $C$ . Recall that the (closed) points of  $\mathcal{M}(C)_\theta^{ss}$  are in one-to-one correspondence with the  $\theta$ -polystable modules in  $C_\theta^{ss}$ . Furthermore, any  $\theta$ -polystable  $A$ -module is a finite direct sum of Schur modules since any  $\theta$ -stable module is Schur. As a direct consequence of these general results, we have:

**Proposition 20.** *A Schur-representation-finite algebra is MF.*

Next, we prove:

**Theorem 21.** *A tame bound quiver algebra is Schur-representation-finite if and only if it is MF.*

*Proof.* The implication “ $\implies$ ” was proved above for the general case.

Now, let  $A = KQ/I$  be a tame bound quiver algebra with the MF property. Assume for a contradiction that there is a dimension vector  $\mathbf{d}$  of  $A$  and an irreducible component  $C \subseteq \text{mod}(A, \mathbf{d})$  that contains infinitely many Schur  $A$ -modules. We immediately deduce from this that  $C$  is not an orbit closure (see for example Lemma 24).

From Crawley-Boevey’s Theorem D in [CB88], we know that all, except finitely many, Schur modules in  $C$  are homogeneous. So, let  $M \in C$  be a homogeneous Schur  $A$ -module. Then,  $C_{\theta^s M}^s$  is a non-empty  $\text{GL}(\mathbf{d})$ -invariant open subset of  $C$  by Lemma 11. Moreover,  $C_{\theta^s M}^s$  must be an infinite disjoint union of orbits since otherwise  $C$  would be an orbit closure. Consequently,  $\dim \mathcal{M}(C)_{\theta^s M}^{ss} \geq 1$  which is in contradiction to  $A$  being MF.  $\square$

**Corollary 22.** *A string algebra  $KQ/I$  is MF if and only if every subquiver  $L \subseteq Q$  of type  $\tilde{A}_n$  contains a relation from  $I$ .*

*Proof.* If  $Q$  contains a type  $\tilde{A}_n$  subquiver  $L$  in which there is no relation, then the family of band modules which are supported precisely on  $L$  and one dimensional at each vertex shows that  $KQ/I$  is not Schur-representation-finite and thus not MF.

On the other hand, if every subquiver of type  $\tilde{A}_n$  in  $Q$  contains a relation, then any band  $B$  must traverse some vertex more than once. H. Krause’s computation of morphisms between band modules (see conditions (H1)-(H4) of [Kra91, p.193]) shows that each band module  $M(B, \lambda)$  associated to  $B$  admits a nilpotent endomorphism. Thus  $KQ/I$  has no Schur band modules and must be Schur-representation-finite, so it is also MF.  $\square$

## 5. OPEN QUESTIONS ON DO ALGEBRAS

In this section, we examine some classes of algebras where an infinite family of representations of the same dimension vector  $\mathbf{d}$  can be used to find a component of  $\text{mod}(A, \mathbf{d})$  without a dense orbit. Our focus on these particular classes is motivated by recent work of Bongartz and Ringel on minimal representation-infinite algebras. It turns out that, any minimal representation-infinite, finite-dimensional algebra is in one of the following three categories (see the Introduction of [Rin11]):

- (1) Algebras with a non-distributive ideal lattice;
- (2) Algebras with a “good” universal cover that contains a convex subcategory which is tame concealed of type  $\tilde{D}_n, \tilde{E}_6, \tilde{E}_7$ , or  $\tilde{E}_8$ ;
- (3) Algebras with a “good” universal cover, but in which all finite convex subcategories are representation finite.

In fact, Ringel showed that all algebras in case (3) are string algebras, and explicitly classified them in terms of quivers with relations. One would like to use this categorization as a proof strategy towards identifying the class of finite-dimensional DO algebras. But there is one technical issue which prevents us from reducing the study of DO algebras to ones on this list, which is that the DO property is not as clearly well behaved under quotients as the MF property (cf. Lemma 13) is. So we pose the following question.

**Question 23.** Is it true that any quotient of a DO algebra is DO?

In any case, we will show now that the representation-infinite string algebras, encompassing case (3), and the triangular non-distributive algebras are not DO. But first, let us quickly prove the following simple lemma:

**Lemma 24.** *Let  $A = KQ/I$  be a bound quiver algebra. If  $A$  admits infinitely many Schur  $A$ -modules of the same dimension then  $A$  does not have the DO property.*

*Proof.* By assumption, there exists a dimension vector  $\mathbf{d}$  of  $A$  and an irreducible component  $C$  of  $\text{mod}(A, \mathbf{d})$  such that  $C$  contains at least two non-isomorphic Schur  $A$ -modules. But, this clearly implies that  $C$  is not an orbit closure. Indeed, assume to the contrary that  $C = \overline{\text{GL}(\mathbf{d})M}$  for some  $M \in \text{mod}(A, \mathbf{d})$ . Now, we know that there exists a Schur  $A$ -module  $E$  such that  $E \in \overline{\text{GL}(\mathbf{d})M} \setminus \text{GL}(\mathbf{d})M$  which implies that  $1 > \dim_K \text{End}_A(M)$ , a contradiction.  $\square$

The following proposition comes directly by following Bongartz's argument in [Bon09, p. 3], with the additional assumption that  $A$  is triangular.

**Proposition 25.** *Let  $A$  be a non-distributive triangular algebra. Then  $A$  admits infinitely many Schur representations of the same dimension. Consequently, non-distributive triangular algebras are not DO.*

*Proof.* Since  $A$  is non-distributive, it admits primitive idempotents  $e, f$  (not necessarily distinct) such that  $fAe$  is neither cyclic as a left  $fAf$ -module nor right  $eAe$ -module; in other words, the radical filtration  $(R^i)$  of  $fAe$  as an  $eAe$ - $fAf$ -bimodule admits a step  $R^l/R^{l+1}$  of dimension  $\geq 2$ . Choose some  $v, w \in R^l$  which are linearly independent modulo  $R^{l+1}$ . Then without loss of generality we can mod out (the two-sided ideal generated by)  $R^{l+1}, Jv, vJ, Jw, wJ$  where  $J = \text{rad } A$ .

If  $A$  is triangular, then in particular there are no loops at any vertex of its ordinary quiver  $Q_A$ , so that  $eAe = fAf = K$  and  $e \neq f$ . Consider the family  $V_\lambda = Ae/\langle v - \lambda w \rangle$  for  $\lambda \in K$ . Applying  $\text{Hom}_A(-, V_\lambda)$  to the quotient, we get  $0 \rightarrow \text{End}_A(V_\lambda) \rightarrow \text{Hom}_A(Ae, V_\lambda)$ . But  $Ae$  is projective, so  $\text{Hom}_A(Ae, V_\lambda) = eV_\lambda = eAe = K$ . So  $V_\lambda$  is a Schur  $A$ -module. That  $A$  is not DO now follows from Lemma 24.  $\square$

**Proposition 26.** *Let  $A = KQ/I$  be a representation-infinite string algebra. Then,  $A$  does not have the DO property.*

*Proof.* Since  $A$  is representation-infinite, we know that there are indecomposable band modules. Among the dimension vectors  $\mathbf{d}$  of  $A$  for which there are infinitely many  $\mathbf{d}$ -dimensional indecomposable band modules, we choose a  $\mathbf{d}$  with  $\sum_{i \in Q_0} \mathbf{d}(i)$  minimal.

Assume to the contrary that  $A$  is DO. Let  $C \subseteq \text{mod}(A, \mathbf{d})$  be an irreducible component that contains (infinitely many) band indecomposable  $A$ -modules. Write  $C = \overline{\text{GL}(\mathbf{d})M_0}$  and denote by  $s$  the number of string indecomposable modules occurring in a direct sum decomposition of  $M_0$  into indecomposable modules.

Now, let  $M \in C$  be a band indecomposable module. Then, we have:

$$\dim_K M = \sum_{a \in Q_1} \text{rank } M(a) \leq \sum_{a \in Q_1} \text{rank } M_0(a) = \dim_K M_0 - s.$$

Consequently,  $s = 0$  and, in fact, due to the minimality of  $\mathbf{d}$ ,  $M_0$  is a band module of the form  $M(B, \lambda^0)$  with  $\lambda^0 \in K^*$ . Now, consider the morphism  $\phi : \mathrm{GL}(\mathbf{d}) \times K^* \rightarrow \mathrm{mod}(A, \mathbf{d})$  defined by  $\phi((g, \lambda)) = gM(B, \lambda)$ . Its image is irreducible, and so, its closure must be  $C$ . In particular,  $M(B, \lambda) \in \overline{\mathrm{GL}(\mathbf{d})M(B, \lambda^0)}$  for all  $\lambda \in K^*$  and since  $M(B, \lambda) \not\cong M(B, \lambda^0)$ ,  $\forall \lambda \neq \lambda^0$ , we deduce that  $\dim_K \mathrm{End}_A(M(B, \lambda)) > \dim_K \mathrm{End}_A(M(B, \lambda^0))$ ,  $\forall \lambda \neq \lambda^0$ . But, this is a contradiction since the endomorphism algebras  $\mathrm{End}_A(M(B, \lambda))$ ,  $\lambda \in K^*$ , have the same dimension (see for example [Kra91]).  $\square$

We end this section with the following conjecture put forth by Weyman:

**Conjecture 27 (Dense Orbit Conjecture (DOC)).** *For a triangular bound quiver algebra  $A = KQ/I$ , the following statements are equivalent:*

- (i) *A is representation-finite;*
- (ii) *for any ideal I of A, the algebra  $A/I$  is DO.*

*Remark 28.* (a) The implication (i)  $\implies$  (ii) is straightforward. The difficult part is to prove the other implication. According to Propositions 25 and 26, to complete the proof of the DOC, it remains to show that a triangular minimal representation-infinite algebra in case (2) is not DO.

(b) One of our main results, Theorem 2, shows that the DOC does not hold true for non-triangular algebras.

## 6. OPEN QUESTIONS ON MF ALGEBRAS

In this final section, we list a series of conjectures on Schur-representation-tame algebras which are intimately related to the class of MF algebras. Let us recall the definition of a Schur-representation-tame algebra from [BD10].

**Definition 29.** A finite-dimensional algebra is said to be *Schur-representation-tame* if, in each dimension vector, all Schur  $A$ -modules, except finitely many, belong to a finite number of 1-parameter families.

We have the following conjectural dichotomy:

**Conjecture 30.** *Let A be a finite-dimensional algebra.*

- (i) *(Ringel's Dichotomy Conjecture) A is either Schur-representation-tame or strictly wild.*
- (ii) *(due to Yang Han) A is strictly wild if and only if A has a wild tilted algebra as a factor.*

Recall that an algebra  $A$  is said to be strictly wild if for every finite-dimensional algebra  $\Lambda$  there exists a fully faithful exact  $K$ -linear functor  $F : \mathrm{mod}(\Lambda) \rightarrow \mathrm{mod}(A)$  (for more details see [SS07b, p. 273 and Lemma XIX.1.5]).

**Example 31.** It follows from the work of Brüstle-de la Peña-Skowroński [BdlPS11] that the strongly simply connected algebras satisfy the conjecture above.

The next conjecture is inspired by Crawley-Boevey's theorem on homogeneous indecomposable modules over tame algebras (see [CB88]):



**Conjecture 32.** *Let  $A$  be a Schur-representation-tame algebra. Then, for each dimension vector  $\mathbf{d}$ , all  $\mathbf{d}$ -dimensional Schur  $A$ -modules  $M$ , except finitely many, are homogeneous, i.e.  $\tau M \simeq M$ .*

Now, we have:

**Proposition 33.** *Let  $A = KQ/I$  be a bound quiver algebra that satisfies Conjectures 30 and 32. If  $A$  is MF then it is Schur-representation-finite.*

*Proof.* Let us assume that  $A$  is MF. Then, we know from Lemma 13 that any of its factors is MF, i.e. for any factor of  $A$ , all of its moduli spaces of modules are zero-dimensional. On the other hand, we know that any wild tilted algebra has moduli spaces of arbitrarily large dimension (see [Chi11]). Consequently,  $A$  has to be Schur-representation-tame.

Now, let us assume that  $A$  is not Schur-representation-finite. Then, there exists a dimension vector  $\mathbf{d}$  and an irreducible component  $C \subseteq \text{mod}(A, \mathbf{d})$  such that  $C$  contains infinitely many Schur  $A$ -modules. In particular,  $C$  contains a homogeneous Schur  $A$ -module  $M$ . It now follows from Lemma 11 that  $M \in C_{\theta_M}^s$  and so  $\mathcal{M}(C)_{\theta_M}^{ss}$  is just a point. From this we deduce that  $C_{\theta_M}^s = \text{GL}(\mathbf{d})M$  and hence  $C = \overline{\text{GL}(\mathbf{d})M}$ . But then, for any other Schur module  $M' \in C$ , we would have  $1 = \dim_K \text{End}_A(M') > \dim_K \text{End}_A(M)$  which is a contradiction. Consequently,  $A$  has to be Schur-representation-finite.  $\square$

Finally, we state another conjecture of Weyman:

**Conjecture 34 (Multiplicity Free Conjecture (MFC)).** *A bound quiver algebra is Schur-representation-finite if and only if it is MF.*

*Remark 35.* Note that according to Proposition 33, to prove the MFC, which is a statement about Schur-representation-finite algebras, one needs to understand first the larger class of Schur-representation-tame algebras.

**Question 36.** Do there exist wild, Schur-representation-finite algebras of finite global dimension?

It has been pointed out to us by Geiss and Schröer that preprojective algebras of Dynkin quivers, which generally are wild and have *infinite* global dimension, are Schur-representation-finite because every Schur module is rigid.

**Question 37.** What are the representation-infinite algebras which are MF?

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